Lecture 30

11.10 - Taylor and Maclauren Series

In the previous lecture, we used geometric series and differentiation and integration to find representations of functions as power series...
However, only a small class of functions can be found in this way (e.g., we can't do this for $\sin x$ or e^x). So, what do we do more generally?

<u>Def</u>: We say $f(x)$ <u>has a power series expansion</u> at a if

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n, \quad |x-a|<R$$

for some $R>0$.

So, this leaves us with some questions to ask:

1) If $f(x)$ has a power series expansion at a, can we tell what it is?

2) For which values of x does $f(x)$ and its power series coincide?

Let's begin by trying to answer the first question.

Taylor Series

Def: If $f(x)$ is a function with infinitely many derivatives at a, the Taylor Series of $f(x)$ about a is the series:

If $a=0$, we call it the Maclaurin Series

Why should we expect this to be the correct power series? Look at derivatives:

If $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Then,

0: $T(a) = f(a) + f'(a)(a-a) + \frac{f''(a)}{2!}(a-a)^2 + \cdots = f(a)$

1: $T'(x) = f'(a) + f''(a)(x-a) + \frac{f'''(a)}{2}(x-a)^2 + \cdots = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1}$

$T'(a) = f'(a)$

2: $T''(x) = f''(a) + f'''(a)(x-a) + \frac{f^{(4)}(a)}{2}(x-a)^2 + \cdots = \sum_{n=2}^{\infty} \frac{f^{(n)}(a)}{(n-2)!}(x-a)^{n-2}$

$T''(a) = f''(a)$

and so on...

Thus $T^{(n)}(a) = f^{(n)}(a)$ for all n.

Let's do examples of finding these series:

Ex: Find the Taylor Series at $a=0$ (i.e., the Maclauren Series) for $f(x)=e^x$.

Ex: Find the Taylor Series at $a=0$ for

$f(x)=$

Ex: Using our methods from last class, since

$$(\ln x)' = \frac{1}{x} = \frac{1}{1-[-(x-1)]},$$

we can find a power series representation, centered at $a=1$, of $\ln x$ as:

$$\ln x = \sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^{n+1}}{n+1}$$

Find the Taylor series centered at $a=1$ for $f(x) = \ln x$.

How does it compare to the above series?

Theorem: If $f(x)$ has a power series expansion at a, i.e., if

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{for } |x-a|<R$$

for some $R>0$, then that power series is the Taylor series of f at a. In particular, we have that

$$c_n = \frac{f^{(n)}(a)}{n!} \quad \text{and} \quad f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

What this theorem is saying is if the function has a power series, then that series is the Taylor series. This is the answer to the first question.

Now, what about the other way? When is $f(x)$ actually equal to ~~its~~ its power series? (This is our second question.)

Let $T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$.

This is the <u>n^{th} Taylor Polynomial of f at a</u>. These are the partial sums of the Taylor Series, so for any value of x such that $T_n(x) \longrightarrow f(x)$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The <u>remainder</u> of the n^{th} Taylor polynomial is

$$R_n(x) = f(x) - T_n(x).$$

The theorem we've been looking for is:

<u>Theorem</u>: Let $f(x)$, $T_n(x)$, and $R_n(x)$ be as above. If

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{for } |x-a| < R,$$

then f is equal to the sum of its Taylor series on $|x-a| < R$.

How do we compute $\lim_{n\to\infty} R_n(x)$?

Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then $R_n(x)$ satisfies

Ex (Taylor's inequality for e^x)

Ex (Taylor's inequality for)

Ex: Prove that e^x is equal to its Maclauren series for all x.

Ex: Compute the limit

$$\lim_{x \to 0} \frac{\cos(x^5) - 1}{x^{10}}$$